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Composition of Non-circular Compression Rings with Optimal Behaviour in Radial Tensile Roofs

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Abstract

Membrane and cable roofs tensioned in radial direction can balance their internal forces with a compression ring. Usually these rings have a circular shape. In an ideal state, the relationship between the load and the curvature causes the ring to resist only a constant axial force, without bending moments. This is an optimal state for sizing. But the circular shape doesn't always coincide with the geometry of the space under the roof and it's necessary to propose another shape for the compression ring. Rings without circular geometry subjected to a centripetal and uniform distributed load must resist axial forces with non-constant value and significant bending moments as well, which makes it difficult to find the optimal sizing. This article shows a system to compose non-circular compression rings subjected to centripetal and variable distributed loads. The formulation of these loads depends on the variation of the curvature at each point of the ring, regarding the same relationship between the load and the curvature, so that the rings only have to resist a uniform axial force without bending moments. These rings are made of combinations of circular and elliptical arcs, and they can approximate different polygonal shapes, offering many possibilities to roof noncircular spaces, even irregular, with an optimal structural behaviour. The process to find this system is described as follows. First, the ideal centripetal distributed load is formulated. After, the combination rules are defined by tangency of circular and elliptical arcs to form closed rings, approximate to any possible polygonal geometries.

Keywords

Membrane; cable; tensile roof; compression ring; ellipse; optimization.

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Initial Approach

Since the first examples were built in the 1950's, radial tensile roofs have been improving to become the most usual type in stadiums and other sport centers[1], because any other solution seems to be less efficient and heavier. The pioneers of these structures became aware of the problem of over-consumption of material resources in construction. This provoked a kind of technological revolution to replace conventional structures by new and more efficient systems[2].

As opposed to other types of tensile structures, those of tensioned radius can balance their internal forces without the necessity of external means, such as ground anchors or big r. c. foundations. These structures need only a boundary compression ring to balance its internal forces. It is, exactly, the element of the biggest material consumption of these structures, and therefore, it is susceptible to be optimized, more than any other element. However, many of the discussions about optimizing these structures have been focused on the design of the geometry of tensioned cables[3], ignoring the behaviour of the ring, often circular.

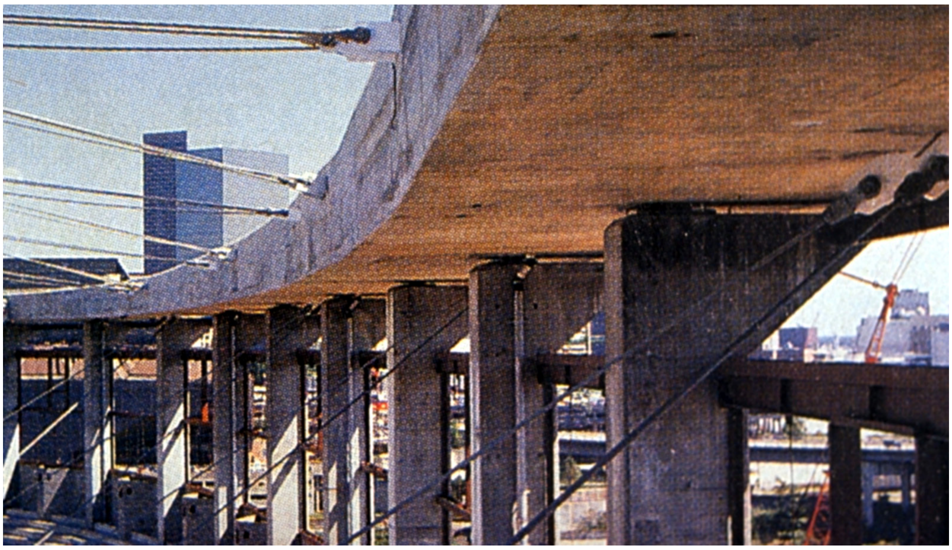

Figure 1.
R. C. compression ring of The Georgia Dome, Atlanta, 1992.
Source: http://www.columbia.edu/cu/gsapp/BT/DOMES/domes.html

This article is part of a broader research to find a system to design tensioned structures on compression flat rings composed of combinations of circumference and ellipse arches[4] in which the conditions for the sizing of its tough section may be optimal, at least in an initial state of load. This means constant axial force throughout its length and lack of significant bending moments. I may call this perfect initial behaviour.

This design system consists of two different parts: the composition of compression rings of initial perfect behaviour, based on the relationship between load and curvature; and the adequacy of the tension of the membrane for an ideal state of load that, applied on the ring, produces a perfect behaviour on it.

The load on the compression ring in a tensile structure is derived from centripetal reactions transmitted by the membrane or cables forming the tensioned interior of the

1. It is possible to see five decades process of building roofs with radial tensile structures on http://www.columbia.edu/cu/gsapp/BT/DOMES/domes.html

2. "We must start with scientific fundamentals, and that means with the data of experiments and not with assumed axioms predicated only upon the misleading nature of that which only superficially seems to be obvious" (Appelwhite and Fuller, 1975: 14).

3. It is possible to see for example Kawaguchi, Tatemichi and Shan Chen (1999) and Netadovic (2010).

4. The composed rings are better suited, that is, that approximate better to the polygonal geometries than the simple circular rings, or even, than the simple elliptical, especially if polygons are not regular.

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structure. These reactions have the direction of the tangent to the trajectory of the membrane or the cables in their joint with the ring. It is possible to decompose it in two directions, one transversal and another one in the plane of the ring. The latter is absorbed by the ring, allowing the reactions to equilibrate themselves inside the structure. Therefore it is said that such structures are autonomous.

The initial state of load on the ring proceeds from the reactions of the membrane or cables subjected to pre-stressing, and to its self weight, usually negligible. Since the membrane has continuous contact with the ring, the reaction on it is a distributed load. However, in a structure of tensioned radius made of cables, the reactions are point loads on the ring. If local bending arising from the incidence of the tensioned radius over the ring is discounted, then the effect of such punctual loads is very similar to that of a distributed load. This article only deals with the design of the rings subjected to distributed loads.

Ideal Distributed Load on the Ring

The ideal load applied to the ring in its plane, in an initial state, must adapt to its curvature, according to the relationship between load (Q) and curvature (C). Thus, the only significant effort to which it will be subjected will be a compression axial force (N) with constant value.

$$N = \frac{Q}{C} \qquad \text{or} \qquad N = Q \cdot R \qquad \text{[1a and 1b]}$$

...where R is the radius, inverse of the curvature.

According to such relationship, the load value (Q) has to modify in each point of the ring as does so curvature (C). In other words, if in a circumference with initial perfect behaviour, the load may be centripetal (pointing to a single center) and uniform, like its curvature, in an ellipse, whose curvature varies at each point and whose radius also have different centers at each point, the load (Q) should be variable in value and direction (centripetal but multi-center) at each point.

To modify the load value (Q) along the perimeter of an elliptical ring there are two possibilities: The first one, if the angle β of incidence of the membrane or the cable with the ring in its transversal plane is the same in the whole the perimeter, modifying the tension of the membrane so that the reaction at each point (x, y) coincides with that defined by the law of charge Q(x), coincident with the function of curvature C(x). The second one, if the tension in the radial direction is uniform across the membrane or cables, modifying the angle of incidence of these on the ring in its transversal plane. It is interesting to imagine, for example, a conoid of tensioned membrane inscribed in an elliptical ring of compression in which, varying the angle of incidence at each point, it would have been adequate the geometry of the membrane until get an ideal state of load in the ring that would only produce a constant axial force, without bending moments.

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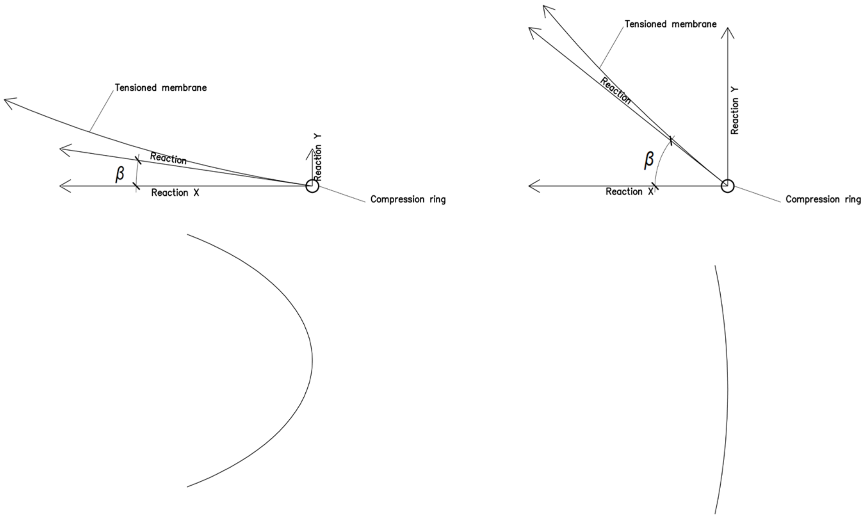

Figures 2a, 2b. Decomposed reaction of the membrane on the ring according two different angles of incidence β.

Then, from the curvature function[5] C(x) of any function y = f(x), formulated as follows:

$$C = \frac{y''\cdot x' - x''y'}{\sqrt{\left(x'^2 + y'^2\right)^3}} \qquad [2]$$

...it is possible to formulate the load function Q(x) as follows:

$$Q(x) = N \cdot C(x) \qquad [3]$$

In the particular case of an ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad [4]$$

Substituting in the above formula x y x' y' x" y" by the function [eq. 5a] , its first and its second derivatives [eq. 5b and 5c], written in the parametric form,

$$x = a \cdot \cos\psi \qquad x' = -a \cdot sen\psi \qquad x'' = -a \cdot \cos\psi$$
$$y = b \cdot sen\psi \qquad y' = b \cdot \cos\psi \qquad y'' = -b \cdot sen\psi$$

[5a; 5b; 5c]

...within which a and b are the major and minor semi-axes, and ψ the angle or reduced latitude of one point of the ellipse, projected in its main circumference, as shown in the figure below,

5. Curvature function from http://mathworld.wolfram.com/Curvature.html

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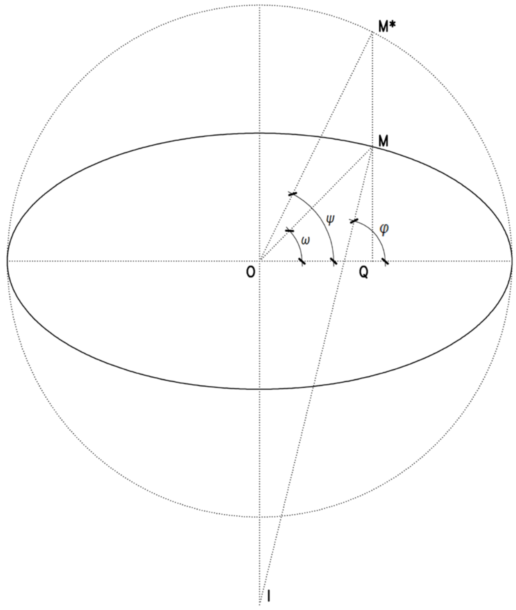

Figure 3.
Geocentric latitude (ω), geodetic (φ) and reduced (ψ) of a point M of the ellipse[6].

…we obtain the curvature function C(ψ) of the ellipse, and therefore also the load function[7] Q(ψ) for a constant value of compression axial force (N) in the ring.

$$C(\psi)=\frac{ab}{\sqrt{\left(a^2sen^2\psi+b^2\cos^2\psi\right)^3}} \quad \text{and} \quad Q(\psi)=\frac{Nab}{\sqrt{\left(a^2sen^2\psi+b^2\cos^2\psi\right)^3}} \qquad [6a;\ 6b]$$

To calculate the minimum and maximum values of Q, simply replace ψ=90° and ψ=0° respectively in eq. 6b;

$$Q_{min}=Q(90°)=\frac{Nb}{a^2} \quad \text{and} \quad Q_{max}=Q(0°)=\frac{Na}{b^2} \qquad [7a;\ 7b]$$

This non-uniform distributed load is formed by an infinite number of vectors whose scalar value may be expressed as follows:

$$Q(x)\cdot dx \quad \text{or} \quad Q(\psi)\cdot d\psi \qquad [8a;\ 8b]$$

...depending on whether it is formulated in terms of x of ψ.

The direction of these vectors is the same as that of the radius of curvature, the normal at each point of the ellipse, namely, the angle φ or geodetic latitude, and it is possible to calculate it from:

$$tg\psi=\frac{b}{a}tg\varphi \qquad [9]$$

To calculate in a straightforward manner the value of the angle φ from the equation of the ellipse in a Cartesian form [eq. 4], first of all, it is necessary to find the gradient of the tangent line and, then, the normal at a point, namely, its first derivative at a point [eq.

6. These latitudes are used to situate any point of the ellipse through different angular reference systems. All three have their zero on the horizontal and rotate counterclockwise to clockwise. The geocentric and reduced systems have a single reference point located at the intersection of the main axes of the ellipse, which is actually that of the polar coordinate system. However, the geodetic system has no single point of reference, and such point varies in each rotation of the angle (Hernández, 1997: 45).

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10a], and then the inverse of the same derivative [eq. 10b],

$$y' = f'(x) = \pm \frac{bx}{a^2\sqrt{1-\frac{x^2}{a^2}}} \qquad \ldots\text{and then}\ldots \qquad \frac{-1}{y'} = \frac{-1}{f'(x)} = \frac{-a^2\sqrt{1-\frac{x^2}{a^2}}}{\pm bx}$$

[10a; 10b]

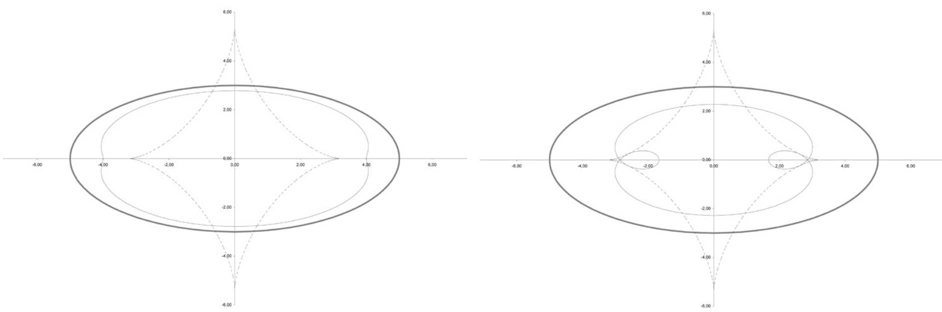
...about which we know that it is equal to the tangent of the angle that forms with the horizontal, that is,

$$\varphi = arctg\left(\frac{-1}{f'(x)}\right)$$

[11]

From the infinite set of vectors, its outer contour can be drawn, that is, the line joining all the end points of these, and thus, a graphical of ideal distributed load function can be represented. To find the coordinates of the points that form this graphic, we need to subtract the scalar value of the load [eq. 6b] in the direction of the vector [eq. 11] to the coordinate (x, y) of each point of the ellipse.

This representation of the ideal distributed load on the ellipse depending on its curvature can be misleading, as two different units are drawn on the same graph scale: that of the length and that of the load. This means that the load value is represented as a length according to an arbitrary graphic scale.



Figures 4a, 4b.
Two identical ellipses with its evolute[7] and one same function for ideal loads represented with two different graphic scales.

An elliptical ring subjected to such loads only suffers deformation as uniform axial as a shortening around the perimeter. This means that the strained ring will remain an ellipse with the same proportions. This shortening will be greater the greater the load factor or the lower will be its resistant section.

This optimal behaviour is relatively easy to prove by composing a calculation model of an ellipse discretized into n segments as a division of the ellipse in sectors with a same inter-angle, and introducing at each one the average value of ideal distributed load corresponding to each sector of the ellipse. The higher density and precision of the discretization, that is, the better the approximation of the ellipse, defined by straight segments, to the real ellipse, the lower the value of the resultant bending moments, because the calculation error will be lesser.

7. Formulated by author.

8. The evolute is the line containing all centers of radius of curvature of a curve. In the case of the ellipse, it islike an asteroid and is formulated as follows:

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

(from html http://mathworld.wolfram.com/EllipseEvolute.html).

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Rings Composition

After defining the function of the ideal load on elliptical rings, it is possible to compose rings by the combination of elliptical and circumference arches, as long as these combinations, that are made not only of arches, but also of ideal distributed loads associated to each arch, satisfying the necessary condition for a perfect behaviour, namely, when the value of the load is function of the curvature on each point. From this condition, as long as the tensioned element is a membrane or distributed cables, that is, that the load on the ring is distributed, result continuous curves, ie, without sudden changes in trajectory.

To reach continuity in the curvature, the joint between two arches must be by tangency, coinciding center and radius of curvature in both arches.

Then, from any point (x, y) of an ellipse, from which it is possible to calculate the geodesic angle φ [eq. 11], the radius of curvature R(x) and the center of such radius on the evolute, one osculating circle[9] is drawn, tangent and of an equal curvature to the ellipse at that point, resulting in a splice without any discontinuity in the path of the line. At this point, the value and direction of a centripetal load with the same factor are identical. And the axial force resulting in the two curves is also the same, the ellipse and the circle, and this is, after all, what avoids any bending moment, and therefore, what allows that the sizing conditions of the resistant section to be optimal.

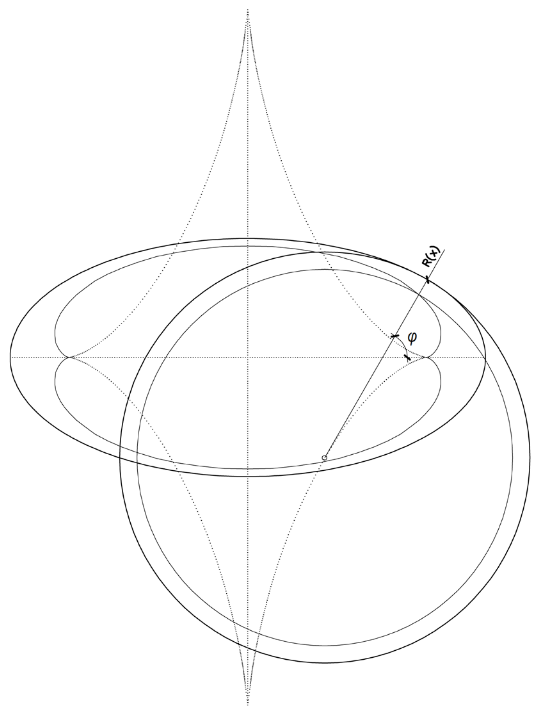

Figure 5.
Tangency between an ellipse and its osculating circle at a point of a geodetic angle φ and radius of curvature R(x).

9. The osculating circle is one that is tangent to the ellipse at a point and also has the same curvature. There is only one osculating circle for each point of the ellipse. The set of all the centers of osculating circles of an ellipse is located in the evolute.

Thus, to compose a ring it is necessary to chain stretches of ellipse arc and/or tangent circumference to form closed figures, of which, the simplest are those formed by two arcs of a same circumference combined with two arches of the same ellipse.

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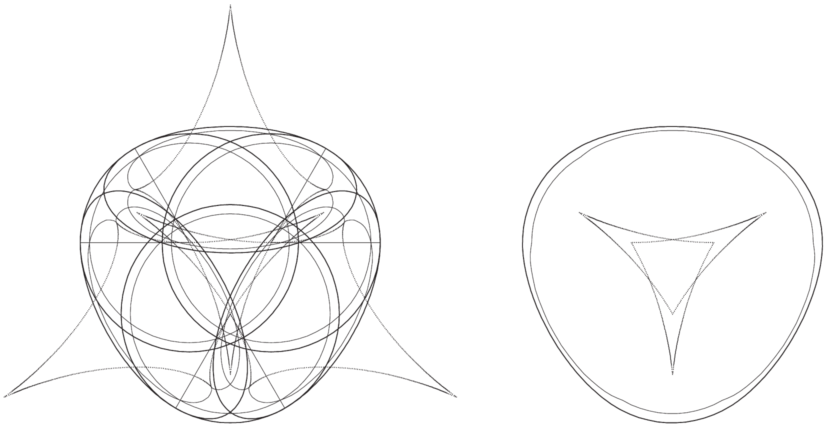
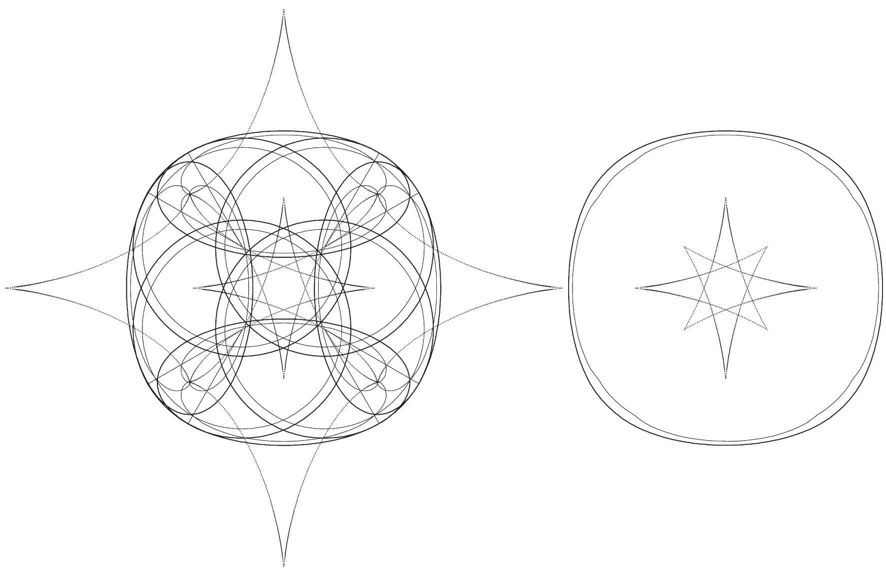


Figures 6a, 6b. Composition of a ring formed by two arches of a same ellipse and two arches of a same osculating circle corresponding to a geodetic angle φ.

According this system, it is possible to compose rings approximate to any regular polygons with three, four or more sides. Rings approximate to polygons with too many sides will look like a circumference, because the distance between the side of the polygon and arch of the polygon becomes smaller the more sides it has.



Figures 6a, 6b. Composition of rings approximate to regular polygons of three sides.

It is also possible to compose rings only by joining ellipse arches, as long as the geodetic angle φ of the ellipse at the point of tangency is equal to the half of angle α between the sides of the regular polygon that approximation. Thus, for composed rings approximate to an equilateral triangle, the tangency will be at φ=30°, for one approximate to a square in φ=45 °, and so on, because the normal to the ellipse in the point of tangency coincides with the bisector of the angle α between the sides of the polygon.

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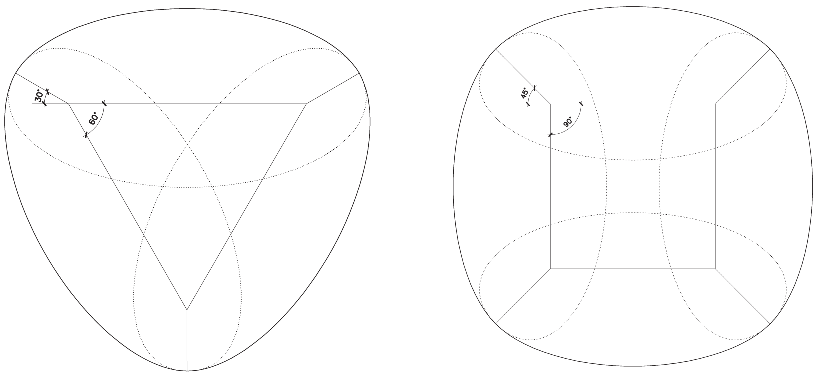
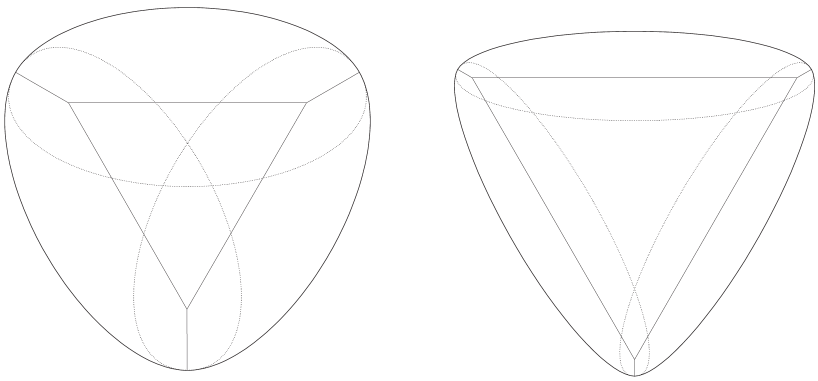
Composed rings only by ellipse arcs allow a better approximation to the polygon, because the radius of curvature at the point of tangency, which is the distance to the vertex of the polygon, is smaller for a same ellipse than when it is combined with circumferences.



Figures 9a, 9b.
Rings composed of elliptical arcs only, approximated to an equilateral triangle and a square.

The greater the eccentricity[10] of the ellipse, the higher also the approximation of the ring to the polygon, although it means that the arches of the ellipse, having lesser curvature, when applying a distributed centripetal load, the compression axial force increases significantly.



Figures 10a, 10b.
The ring on the right has a higher approximation to the triangle than the ring of the left, because it is formed by ellipses of greater eccentricity.

Conclusion

The deductive process to obtain the composition rules of non-circular rings is as follows: optimum conditions for sizing the compression ring in radial tensile roof have been defined, the ideal function load-curvature of elliptical rings has been formulated, and conditions of tangency between circumference and elliptical arcs and the continuity between both of the ideal function of load-curvature have been defined. At last, composition rules of rings approximate to simple polygons and the possible variations of the eccentricity between the ring and the polygon have been set.

Following these rules of composition, it is possible to compose rings approximated to any irregular polygons too, by combining arches of different circumferences and ellipses.

10. The eccentricity of the ellipse ε refers to the relationship between its main axes, and is calculated as follows:

$$\varepsilon = \frac{\sqrt{a^2 - b^2}}{a}$$

from http://mathworld.wolfram.com/Eccentricity.html

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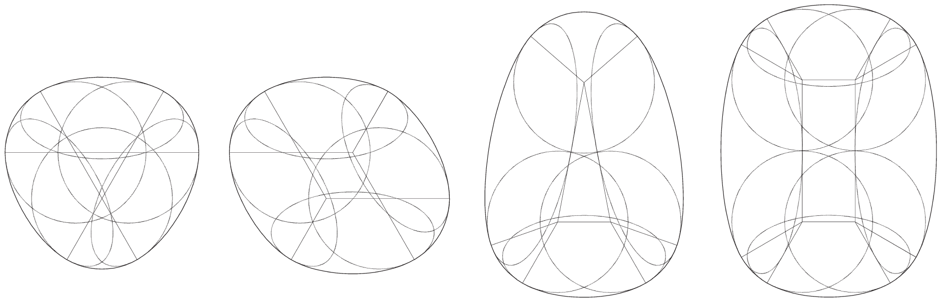
Some examples are shown bellow.



Figures 11a, 11b, 11c, 11d. Four different compression rings composed according rules for optimal behaviour.

This opens the possibility to design the roofing system of regular and irregular polygonal geometries by recurring to tensioned structures of radial distribution without having to renounce to the optimization of the sizing of their compression ring.

It will be necessary to follow this research to formulate rules that define the relationship between the geometry and the membrane or cable tension, and the reaction on the compression ring, that is, the control of the load on the ring, defined in this research article.

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